EXAM MECHATRONICS

WEDNESDAY 20 JANUARY 2016, 9.00 - 12.00 h.

Name: ______________________________ StudentID: ____________

- This exam consists of 24 pages with 6 open questions. Check if you have all pages.
- The answers to each question (including motivation) have to be placed in the answer boxes.
- Please write your name and student number in all pages. The exercises will be collected separately.
- If you like, you can use and add additional paper, which needs to include your name and student number. Please provide separate papers for separate exercises. Hand in the exercises on separate piles.
- You can earn a maximum of 100 points at the exam. The amount of points spread over the exercises is 110 points, i.e., there are 10 bonus points to be earned.
- You will only get a grade if you have finalized the practical.
- This is a CLOSED book exam.
- Good luck!

Preliminaries

Across and Through variable table

Table 1.2. Ideal system elements (linear)

System type	Mechanical translational	Mechancial rotational	Electrical	Fluid	Thermal
A-type variable	Velocity, v	Velocity, Ω	Voltage, e	Pressure, P	Temperature, T
A-type element	Mass, m	Mass moment of inertia, J	Capacitor, C	Fluid Capacitor, C_f	Thermal capacitor, C_h
Elemental equations	$F = m\frac{dv}{dt}$	$T = J\frac{d\Omega}{dt}$	$i = C\frac{de}{dt}$	$Q_f = C_f\frac{dP}{dt}$	$Q_h = C_h\frac{dT}{dt}$
Energy stored	Kinetic	Kinetic	Electric field	Potential	Thermal
Energy equations	$\mathscr{E}_k = \frac{1}{2}mv^2$	$\mathscr{E}_k = \frac{1}{2}J\Omega^2$	$\mathscr{E}_e = \frac{1}{2}Ce^2$	$\mathscr{E}_p = \frac{1}{2}C_f P^2$	$\mathscr{E}_t = \frac{1}{2}C_h T^2$
T-type variable	Force, F	Torque, T	Current, i	Fluid flow rate, Q_f	Heat flow rate, Q_h
T-type element	Compliance, $1/k$	Compliance, $1/K$	Inductor, L	Inertor, I	None
Elemental equations	$v = \frac{1}{k}\frac{dF}{dt}$	$\Omega = \frac{1}{K}\frac{dT}{dt}$	$e = L\frac{di}{dt}$	$P = I\frac{dQ_f}{dt}$	
Energy stored	Potential	Potential	Magnetic field	Kinetic	
Energy equations	$\mathscr{E}_P = \frac{1}{2k}F^2$	$\mathscr{E}_P = \frac{1}{2K}T^2$	$\mathscr{E}_m = \frac{1}{2}Li^2$	$\mathscr{E}_k = \frac{1}{2}IQ_f^2$	
D-type element	Damper, b	Rotational damper, B	Resistor, R	Fluid resistor, R_f	Thermal resistor, R_h
Elemental equations	$F = bv$	$T = B\Omega$	$i = \frac{1}{R}e$	$Q_f = \frac{1}{R_f}P$	$Q_h = \frac{1}{R_h}T$
Rate of energy dissipated	$\frac{dE_D}{dt} = Fv = \frac{1}{b}F^2 = bv^2$	$\frac{dE_D}{dt} = T\Omega = \frac{1}{B}T^2 = B\Omega^2$	$\frac{dE_D}{dt} = ie = Ri^2 = \frac{1}{R}e^2$	$\frac{dE_D}{dt} = Q_f P = R_f Q_f^2 = \frac{1}{R_f}P^2$	$\frac{dE_D}{dt} = Q_h$

Note: A-type variable represents a spatial difference across the element.

The other analogy as was treated in Control Engineering, and is useful for Euler-Lagrange modeling.

	Kinetic coenergy	Potential energy	Rayleigh dissipation function
Translation	$T^*(\dot{x}) = \frac{1}{2}m\frac{dx}{dt}^2$	$V(x) = \frac{1}{2}kx^2$	$\mathcal{D}(\dot{x}) = \frac{1}{2}b\frac{dx}{dt}^2$
Rotation	$T^*(\dot{\theta}) = \frac{1}{2}J\frac{d\theta}{dt}^2$	$V(\theta) = \frac{1}{2}k\theta^2$	$\mathcal{D}(\dot{\theta}) = \frac{1}{2}b\frac{d\theta}{dt}^2$
Electric	$T^*(\dot{q}) = \frac{1}{2}L\frac{dq}{dt}^2$	$V(q) = \frac{1}{2C}q^2$	$\mathcal{D}(\dot{q}) = \frac{1}{2}R\frac{dq}{dt}^2$

Canonical forms

The state-space representation for a given transfer function is not unique, i.e., there are infinite-number of possibilities to express a given transfer function in state-space form. However, there are several forms that can be helpful in the design of controller or observer. Let us consider the following general transfer function for single-input single-output system:

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}. \tag{1}$$

For this transfer function, the state-space representation in **canonical observable form** is given by

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} &= \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1}b_0 \\ \vdots \\ b_1 - a_1 b_0 \end{bmatrix} u \\
y &= \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u.
\end{aligned} \tag{2}$$

On the other hand, the state-space representation in the **canonical controllable form** is given by

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \\
y &= \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1}b_0 & \cdots & b_2 - a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u.
\end{aligned} \tag{3}$$

Z-transform.

Denote by $Z\{u(n)\}$ the Z-transform of discrete-time signal $u(n)$ where $n = 0, 1, \ldots$.

- Unit step signal $u(n)$: $Z\{u(n)\} = \dfrac{1}{1-z^{-1}}$
- Time-shifting property: $Z\{u(n-k)\} = z^{-k}U(z)$

Transformation from s-domain to z-domain

- The bilinear transformation: $s \mapsto \dfrac{2}{T}\dfrac{1-z^{-1}}{1+z^{-1}}$
- The backward-Euler transformation: $s \mapsto \dfrac{1}{T}(1-z^{-1})$

Optimal state feedback control design(LQR)

The Riccati equation, that is related to the optimal state feedback, reads as

$$A^T P + PA - PBR^{-1}B^T P + Q = 0, \tag{4}$$

where Q and R are related to the cost function

$$J = \int_0^\infty x^T(\tau)Qx(\tau) + u^T Ru(\tau)d\tau. \tag{5}$$

The optimal state feedback controller is given by $u(t) = -R^{-1}B^T Px(t)$.

State observer design

For a state-space system described by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du, \end{aligned} \tag{6}$$

where x is the actual state and y is the measured signal, a state observer for such system has the following structure

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) \\ \hat{y} &= C\hat{x}, \end{aligned} \tag{7}$$

where $\hat{x}$ is the estimated state and y is the corresponding estimated output.

Transfer function of time delay

For a time delay operator

$$y(t) = u(t - T) \tag{8}$$

where T is the delay time, its Laplace transform is given by

$$\frac{Y(s)}{U(s)} = e^{-sT}. \tag{9}$$

From this transfer function, the corresponding Bode plot is given by unity amplitude for all frequencies and the phase plot is linear with respect to the frequency, i.e.,

$$\phi(\omega) = -\omega T, \tag{10}$$

for all frequencies ω.

The first order Padé approximation of the delay transfer function $e^{-\omega T}$ is given by

$$e^{-\omega T} \approx \frac{1 - \frac{T}{2}s}{1 + \frac{T}{2}s}. \tag{11}$$

Exercises

Exercise 1 (15 points)

Consider the robotic vacuum system iRobot Roomba as shown in figure 1

Figure 1: The iRobot Roomba system

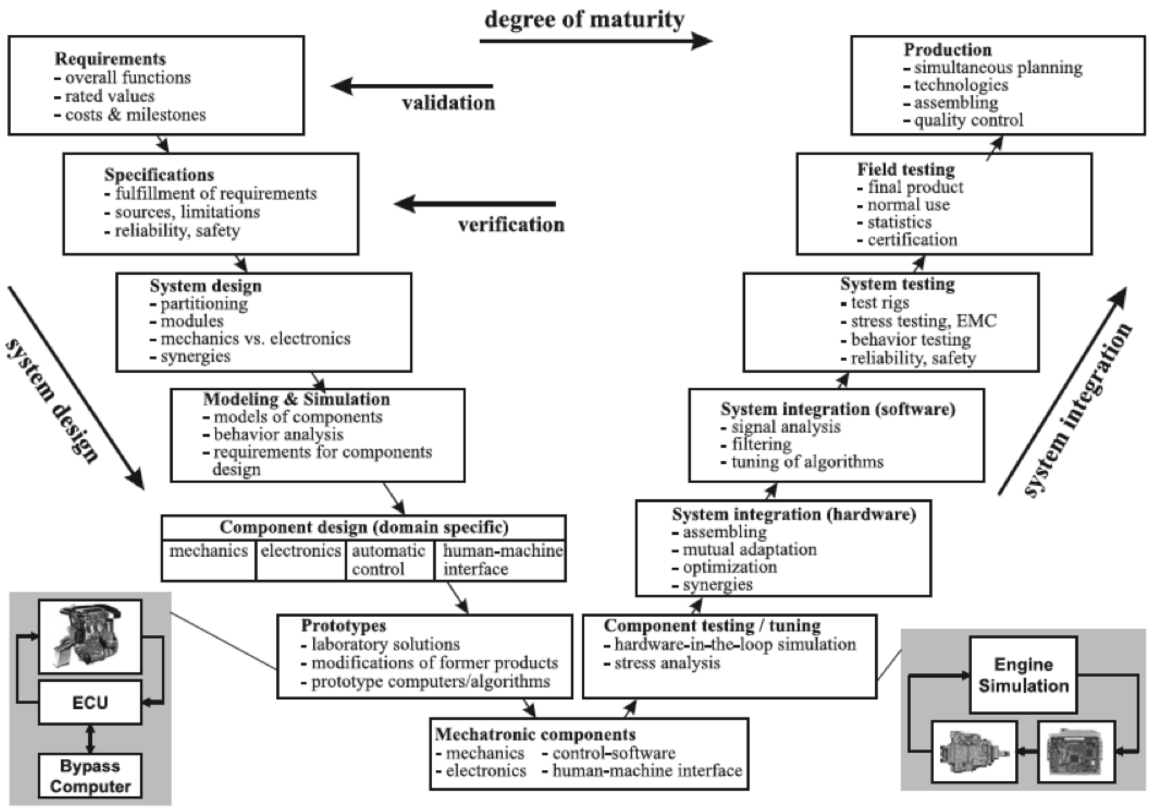
(a) (7 points) Identify two possible user demands, two possible functional requirements, and two possible design parameters.

Solution:

- User demands
  - Cheap
  - Silent
  - Quick
  - Battery autonomy
  - Auto-docking for automatic recharge
  - Possibility to clean several surfaces and corners.
- Functional requirements
  - The capacity of the dust-tank should be of x liters
  - The autonomy of the robot should be of x- hours
  - The lifetime of the components should be of x-years
  - The noise produced by the robot should be of x-dB.
- Design parameters
  - Sensors to detect position and avoid collisions
  - Actuator to ensure proper and efficient cleaning.
  - The outer material should be hard enough to sustain some collisions.
  - Size.

(b) (8 points) Provide the blocks for the Mechatronic design cycle (V-diagram) for the design of the iRobot Roomba system.

Solution: The V-diagram looks like follows



1. User demands and functional requirements
2. Design parameters
3. Overall design with component specifications and integration plan.
4. CAD/CAM/CAE or even mathematical modeling.
5. Design of specific components as the sensors and actuators.
6. Prototyping of the cleaning mechanism and/or the localization interface.
7. Integration of some components and subsystems such as the wheels and steering system.
8. Testing of durability, efficiency, quickness, etc

9-10 Full combination and integration of software and hardware.

11 Test predetermined instructions, cleaning patterns, cleaning capabilities, collision avoidance, etc.

12 Test in real environment. Test robustness of the robot and compare with specifications.

13 Production

Exercise 2 (20 TOTAL+5 bonus points)

Consider a pendulum supported on an inclined plane as shown in Figure 2

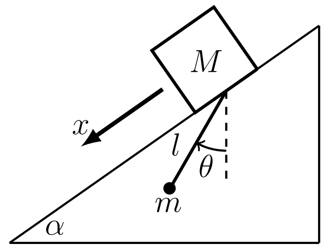


Figure 2: A pendulum supported on an inclined plane

The mass M is free to slide down the frictionless plane inclined at an angle α. A pendulum of length l and mass m hangs from M. (Assume that M extends beyond the side of the inclined plane, so the pendulum can hang down).

(a) (12 points) Find the equations of motion via the Euler-Lagrange method. Recall that the equations of motion are obtained via the formula

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q,\dot{q})\right) - \frac{\partial L}{\partial q}(q,\dot{q}) = -\frac{\partial D}{\partial \dot{q}} + Gu, \tag{12}$$

where L is the Lagrangian, D the Rayleigh dissipation function, and G the input matrix. **Hint:** The position of the block is given by $(x_M, y_M) = (x\cos\alpha, -x\sin\alpha)$, while the position of the pendulum is given by $(x_m, y_m) = (x_M + l\sin\theta, y_M - l\cos\theta)$.

Solution: From the figure, let x be the coordinate of the block M along the plane and θ the angle of the pendulum. The coordinates of the block are

$$(x_M, y_M) = (x\cos\alpha, -x\sin\alpha), \tag{13}$$

while the coordinates of the pendulum are

$$\begin{aligned}(x_m, y_m) &= (x_M + l\sin\theta, y_M - l\cos\theta)\\ &= (x\cos\alpha + l\sin\theta, -x\sin\alpha - l\cos\theta).\end{aligned} \tag{14}$$

To obtain the velocities of the block M and the pendulum m we differentiate the positions obtaining

$$\begin{aligned}v_M^2 &= \dot{x}_M^2 + \dot{y}_M^2 = (\dot{x}\cos\alpha)^2 + (-\dot{x}\sin\alpha)^2\\ &= \dot{x}^2.\\ v_m^2 &= \dot{x}_m^2 + \dot{y}_m^2 = (\dot{x}\cos\alpha + l\dot{\theta}\cos\theta)^2 + (-\dot{x}\sin\alpha + l\dot{\theta}\sin\theta)^2\\ &= \dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}\dot{\theta}l(\cos\alpha\cos\theta - \sin\alpha\sin\theta).\end{aligned} \tag{15}$$

The Kinetic energy is given by

$$\begin{aligned}K &= \frac{1}{2}Mv_M^2 + \frac{1}{2}mv_m^2\\ &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}\dot{\theta}l(\cos\alpha\cos\theta - \sin\alpha\sin\theta)\right)\end{aligned} \tag{16}$$

The Potential energy is given by

$$\begin{aligned} P &= Mgy_M + mgy_m = Mg(-x\sin\alpha) + mg(-x\sin\alpha - l\cos\theta)) \\ &= -(M+m)gx\sin\alpha - mgl\cos\theta. \end{aligned} \tag{17}$$

Then the Lagrangian $L = K - P$ is given by

$$\begin{aligned} L &= K - P \\ &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}\dot{\theta}l(\cos\alpha\cos\theta - \sin\alpha\sin\theta)\right) + (M+m)gx\sin\alpha + mgl\cos\theta. \end{aligned} \tag{18}$$

Now we have the partial derivatives

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}} &= (M+m)\dot{x} + ml\dot{\theta}(\cos\alpha\cos\theta - \sin\alpha\sin\theta) \\ \frac{\partial L}{\partial \dot{\theta}} &= ml^2\dot{\theta} + ml\dot{x}(\cos\alpha\cos\theta - \sin\alpha\sin\theta) \\ \frac{\partial L}{\partial x} &= (M+m)g\sin\alpha \\ \frac{\partial L}{\partial \theta} &= ml\dot{x}\dot{\theta}(-\cos\alpha\sin\theta - \sin\alpha\cos\theta) - mgl\sin\theta, \end{aligned} \tag{19}$$

and the time derivatives

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= (M+m)\ddot{x} + ml\ddot{\theta}(\cos\alpha\cos\theta - \sin\alpha\sin\theta) - ml\dot{\theta}^2(\cos\alpha\sin\theta + \sin\alpha\cos\theta) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= ml^2\ddot{\theta} + ml\ddot{x}(\cos\alpha\cos\theta - \sin\alpha\sin\theta) + ml\dot{x}\dot{\theta}(-\cos\alpha\sin\theta - \sin\alpha\cos\theta). \end{aligned} \tag{20}$$

Therefore the equations of motion read as

$$\begin{aligned} (M+m)\ddot{x} + ml\left(\ddot{\theta}\underbrace{(\cos\alpha\cos\theta - \sin\alpha\sin\theta)}_{\cos(\theta+\alpha)} - \dot{\theta}^2\underbrace{(\cos\alpha\sin\theta + \sin\alpha\cos\theta)}_{\sin(\theta+\alpha)}\right) &= (M+m)g\sin\alpha \\ ml^2\ddot{\theta} + ml\ddot{x}\underbrace{(\cos\alpha\cos\theta - \sin\alpha\sin\theta)}_{\cos(\theta+\alpha)} &= -mgl\sin\theta. \end{aligned} \tag{21}$$

(b) (8 points) Write down the energy storing elements and the physical variables associated to them. Afterwards, indicate which and how many state-space variables are minimally necessary to describe the system provided that we are interested in the position of the pendulum θ. Motivate your answer.

Solution: The two energy storing elements are the masses (of the block and the pendulum). The related physical variables are their velocities v_M and v_m respectively. In principle these two variables would be enough to describe the system, however since we are explicitely asked to also consider the position of the pendulum θ, we conclude that the three minimal variables to describe the system are $(\dot{x}, \theta, \dot{\theta})$.

Another reasoning: since the position is not in the Equations of motion we conclude that the three states $(\dot{x}, \theta, \dot{\theta})$ are enough to describe the system.

(c) (5 bonus points) An equilibrium point corresponds to $\ddot{\theta} = \dot{\theta} = 0$. Assume the the slope is infinitely long. What is the corresponding equilibrium value of θ?

Hint: Substitute $\ddot{\theta} = \dot{\theta} = 0$ into the equations of motion obtained above and try to obtain θ. You can also use your imagination and motivate your answer with physical arguments.

Solution: For the bonus points we have the relations

$$\begin{aligned}
\sin\alpha(\cos\alpha\cos\theta - \sin\alpha\sin\theta) &= -\sin\theta \\
\sin\alpha\cos\alpha\cos\theta - \sin^2\alpha\sin\theta &= -\sin\theta \\
\sin\alpha\cos\alpha\cos\theta &= \sin\theta(1-\sin^2\theta) \\
\sin\alpha\cos\theta &= -\sin\alpha\cos\alpha \\
\tan\alpha &= -\tan\theta.
\end{aligned} \tag{22}$$

Then $\theta = \{-\alpha, -\alpha + \pi\}$.

The physical interpretation is evident...

Exercise 3 (20 TOTAL points)

Suppose you are given a system with transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2}{s^2 + 3s + 4}, \tag{23}$$

where $U(s)$ is the Laplace transform of the input u, and $Y(s)$ is the Laplace transform of the output y.

(a) (2 points) Is the plant stable or unstable? Motivate your answer.

Solution: The plant is stable because its characteristic polynomial has roots

$$s = \frac{-3 \pm \sqrt{-7}}{2}. \tag{24}$$

(b) (2 points) What is the corresponding differential equation of the plant?

Solution:

$$\ddot{y}(t) + 3\dot{y}(t) + 4y(t) = 2u(t). \tag{25}$$

(c) (8 points) Design a PID controller (its transfer function is $C(s) = K_p + sK_d + \frac{1}{s}K_i$) such that the closed-loop system is stable and the corresponding characteristic polynomial has roots at $\{-1, -2, -3\}$, this is $p(s) = (s+1)(s+2)(s+3)$.

Solution: The closed-loop characteristic polynomial can be computed by the numerator of $1+G(s)C(s)$ (the sensitivity transfer function of the closed loop transfer function can equivalently be used). With this we obtain the characteristic

$$s^3 + (3 + 2K_d)s^2 + (4 + 2K_p)s + 2K_i. \tag{26}$$

On the other hand, the required closed-loop characteristic polynomial is

$$(s+1)(s+2)(s+3) = s^3 + 6s^2 + 11s + 6. \tag{27}$$

The gains of the controller are obtained from (26) and (27):

$$s^3 + (3 + 2K_d)s^2 + (4 + 2K_p)s + 2K_i = s^3 + 6s^2 + 11s + 6. \tag{28}$$

Therefore $K_p = \frac{7}{2}$, $K_d = \frac{3}{2}$, $K_i = 3$.

(d) (8 points) Find the corresponding discrete-time controller $C(z)$ related to the controller $C(s)$ obtained in (c). For this you can use either the bilinear or the backward Euler approximation with a sampling time of $T = 2$s. Determine the corresponding difference equation.

Solution:
The controller now reads as

$$C(s) = \frac{7}{2} + \frac{3}{2}s + \frac{3}{s}. \tag{29}$$

- Using the backward Euler approximation we have $s \mapsto \frac{1}{2}(1 - z^{-1})$. Then

$$\begin{aligned} C(z) &= \frac{7}{2} + \frac{3}{4}(1 - z^{-1}) + \frac{6}{1 - z^{-1}} \\ &= \frac{\frac{7}{2}(1 - z^{-1}) + \frac{3}{4}(1 - z^{-1})^2 + 6}{1 - z^{-1}} \\ &= \frac{\frac{7}{2}(1 - z^{-1}) + \frac{3}{4}(1 - 2z^{-1} + z^{-2}) + 6}{1 - z^{-1}} \\ &= \frac{\frac{3}{4}z^{-2} - 5z^{-1} + \frac{41}{4}}{1 - z^{-1}} \end{aligned} \tag{30}$$

  Now recall that for a controller we have the relation $U(z) = C(z)E(z)$ and therefore the corresponding difference equation reads as

$$U(k) - U(k-1) = \frac{41}{4}E(k) - 5E(k-1) + \frac{3}{4}E(k-2). \tag{31}$$

- Using the bilinear approximation we have $s \mapsto \frac{1 - z^{-1}}{1 + z^{-1}}$. Then

$$\begin{aligned} C(z) &= \frac{7}{2} + \frac{3}{2}\frac{1 - z^{-1}}{1 + z^{-1}} + 3\frac{1 + z^{-1}}{1 - z^{-1}} \\ &= \frac{\frac{7}{2}(1 - z^{-2}) + \frac{3}{2}(1 - z^{-1})^2 + 3(1 + z^{-1})^2}{1 - z^{-2}} \\ &= \frac{z^{-2} + 3z^{-1} + 8}{1 - z^{-2}}. \end{aligned} \tag{32}$$

  Now recall that for a controller we have the relation $U(z) = C(z)E(z)$ and therefore the corresponding difference equation reads as

$$U(k) - U(k-2) = 8E(k) + 3E(k-1) + E(k-2) \tag{33}$$

Exercise 4 (30 TOTAL points)

Suppose that a transfer function of a DC motor with a voltage input u to the measurement output y (which, in this case, is the angular displacement of the motor) is given by

$$\frac{Y(s)}{U(s)} = \frac{2}{(s+2)(s+3)}. \tag{34}$$

(a) (5 points) Define the state variables $(x_1, x_2) = (y, \dot{y})$. Write down a state space equation which corresponds to the given transfer function.

Solution: There are many ways to write a state space equation related to the transfer function.

- The canonical controllable form would read as

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \tag{35}$$

- The canonical observable form would read as

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & -6 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \tag{36}$$

- The corresponding differential equation would read as ***(Most probably students will use this form)

$$\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = 2u(t). \tag{37}$$

  In this case defining the state space variables $(x_1, x_2) = (y, \dot{y})$ we would obtain the state space equation

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \tag{38}$$

(b) (10 points) Design an optimal state feedback controller which minimizes the following cost function

$$J = \int_0^\infty 5x_1^2(\tau) + 6x_1(\tau)x_2(\tau) + 3x_2^2(\tau) + 13u^2(\tau)d\tau \tag{39}$$

Solution: Note that

$$Q = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}, \qquad R = 13. \tag{40}$$

Now we have to solve the Riccati equation

$$A^T P + PA - PBR^{-1}B^T P + Q = 0. \tag{41}$$

- In the case of the controllable canonical form we have

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{42}$$

The solution of the Riccati equation reads as

$$P \approx \begin{bmatrix} 1.38 & 0.42 \\ 0.42 & 0.38 \end{bmatrix}. \tag{43}$$

Then the corresponding optimal controller gain is given by

$$u = -Kx = -B^T R^{-1} P x \approx -\begin{bmatrix} 0.033 & 0.03 \end{bmatrix} x. \tag{44}$$

- In the case of the observable canonical form we have

$$A = \begin{bmatrix} 0 & -6 \\ 1 & -5 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \tag{45}$$

The solution of the Riccati equation reads as

$$P \approx \begin{bmatrix} 2.4 & -1.61 \\ -1.61 & 2.16 \end{bmatrix}. \tag{46}$$

Then the corresponding optimal controller gain is given by

$$u = -Kx = -B^T R^{-1} P x \approx -\begin{bmatrix} 0.37 & -0.25 \end{bmatrix} x. \tag{47}$$

- In the third option we have

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \tag{48}$$

The solution of the Riccati equation reads as

$$P \approx \begin{bmatrix} 1.38 & 0.41 \\ 0.41 & 0.38 \end{bmatrix}. \tag{49}$$

Then the corresponding optimal controller gain is given by

$$u = -Kx = -B^T R^{-1} P x \approx -\begin{bmatrix} 0.063 & 0.058 \end{bmatrix} x. \tag{50}$$

In any case, we have that the matrix $(A - BK)$ is Hurwitz and that the function J is minimized.

(c) (2 points) What is the convergence rate of the plant? Motivate your answer.

Solution: Since the eigenvalues of the plant are $-2, -3$, we know that the convergence rate is given by the "slowest" eigenvalue or the one with smaller absolute value. Therefore the convergence rate of the plant is 2.

(d) (10 points) Suppose that the measurement of the angular velocity is not available. Design a state observer that can provide an estimate of the angular velocity with a prescribed (exponential) convergence rate of 25 (e.g., $|x(t) - \hat{x}(t)| \leq |x(0) - \hat{x}(0)| \exp(-25t)$ where $x, \hat{x}$ are the state and the estimated state, respectively).

Solution: Let $L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$. From the desired observer convergence rate let us chose the eigenvalues of the closed loop matrix $A - LC$ at $-25, -25$. This is that the desired closed loop characteristic polynomial reads as $p(\lambda) = \lambda^2 + 50\lambda + 625$. Then

- For the state-space equation written in controllable canonical form we have

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 0 \end{bmatrix}. \tag{51}$$

Therefore

$$A - LC = \begin{bmatrix} -2l_1 & 1 \\ -6 - 2l_2 & -5 \end{bmatrix}, \tag{52}$$

and the corresponding characteristic polynomial is given by

$$\lambda^2 + (5 + 2l_1)\lambda + 10l_1 + 6 + 2l_2. \tag{53}$$

Then from the equality

$$\lambda^2 + (5 + 2l_1)\lambda + 10l_1 + 6 + 2l_2 = \lambda^2 + 50\lambda + 625, \tag{54}$$

we conclude that

$$l_1 = 22.5, \qquad l_2 = 197. \tag{55}$$

- For the state-space equation written in observable canonical form we have

$$A = \begin{bmatrix} 0 & -6 \\ 1 & -5 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 \end{bmatrix}. \tag{56}$$

Therefore

$$A - LC = \begin{bmatrix} 0 & -6 - l_1 \\ 1 & -5 - l_2 \end{bmatrix}, \tag{57}$$

and the corresponding characteristic polynomial is given by

$$\lambda^2 + (5 + 2l_2)\lambda 6 + 1l_1. \tag{58}$$

Then from the equality

$$\lambda^2 + (5 + l_2)\lambda + 6 + l_1 = \lambda^2 + 50\lambda + 625, \tag{59}$$

we conclude that

$$l_1 = 619, \qquad l_2 = 45. \tag{60}$$

- For the third state-space equation we have

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{61}$$

Therefore

$$A - LC = \begin{bmatrix} -l_1 & 1 \\ -6 - l_2 & -5 \end{bmatrix}, \tag{62}$$

and the corresponding characteristic polynomial is given by

$$\lambda^2 + (5 + l_1)\lambda + 5l_1 + 6 + l_2. \tag{63}$$

Then from the equality

$$\lambda^2 + (5 + l_1)\lambda + 5l_1 + 6 + l_2 = \lambda^2 + 50\lambda + 625, \tag{64}$$

we conclude that

$$l_1 = 45, \qquad l_2 = 394. \tag{65}$$

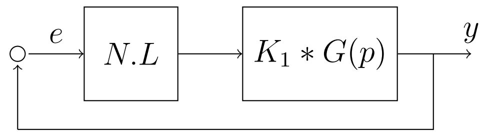
(e) (3 points) Assume you want to implement the optimal state feedback controller using the estimated velocity. How do you do this? Motivate your answer and provide enough justification. **Hint: Recall the separation principle**

Solution: To implement the optimal controller $u = -Kx$ from the estimated variables we just use the estimated variables $\hat{x}$ instead of the original ones. Recall that (available in Nestor) the full closed loop system of the controller error and observer error reads as

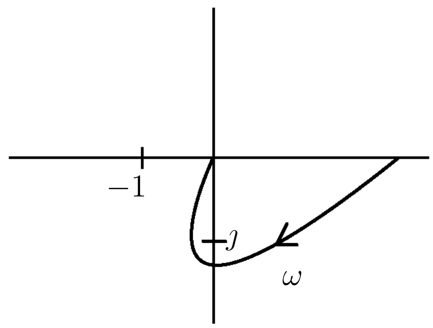
$$\begin{bmatrix} \frac{d\tilde{x}_1}{dt} \\ \frac{d\tilde{x}_2}{dt} \\ \frac{de_1}{dt} \\ \frac{de_2}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}}_{H} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ e_1 \\ e_2 \end{bmatrix}, \tag{66}$$

where $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$ denotes the error of the optimal closed loop system while $e = (e_1, e_2) = (x_1 - \hat{x}_1, x_2 - \hat{x}_2)$ denotes the observer error. Since $A - BK$ and $A - LC$ are *by design* stable, the matrix H is also stable. This implies that all errors (controller and observer) converge to zero.

Exercise 5 (5 TOTAL + 5 bonus points)
In the non-linear system below,



the plant $G(p)$ is stable, and the static non-linearity NL is given by a non-linear sector with boundaries $[0, k]$. The Popov plot from the linear portion is given by



(a) (2 points) Which of the following is/are true, motivate your answer.

1. For $k = -2$ the system is stable
2. For $k = 2$ the system is stable
3. For $k = -1$ the system is unstable
4. For $k = 1$ the system is unstable
5. None of the above.

Solution: For any $k > 0$ the closed-loop system will be stable. Therefore 2. is correct.

(b) (3 points) If $G(p)$ is unstable, can we conclude the same as in (a)? Motivate your answer.

Solution: No, since the Popov criteria provides only sufficient conditions for stability.

(c) (2 bonus points) What is the relation between the Popov plot and the Nyquist plot of a system?

Solution: Given a transfer function $G(s)$, where s is a complex variable, the Nyquist plot corresponds to the plot (in the complex plane) of $G(\jmath\omega) = G_1(\jmath\omega) + \jmath G_2(\jmath\omega)$ while the Popov plot corresponds to the plot (in the complex plane) of $W(\jmath\omega) = G_1(\jmath\omega) + \jmath\omega G_2(\jmath\omega)$.

(d) (3 bonus points) Which is the biggest sector (approximately) to which the non-linearity may belong so that the closed-loop system is globally asymptotically stable? Motivate your answer.

Solution: $k = \infty$ since the Popov criteria is satisfied by a line passing through the origin.

Exercise 6 (10 TOTAL points)

Consider a simplified turbocharger system as shown in Figure 3.

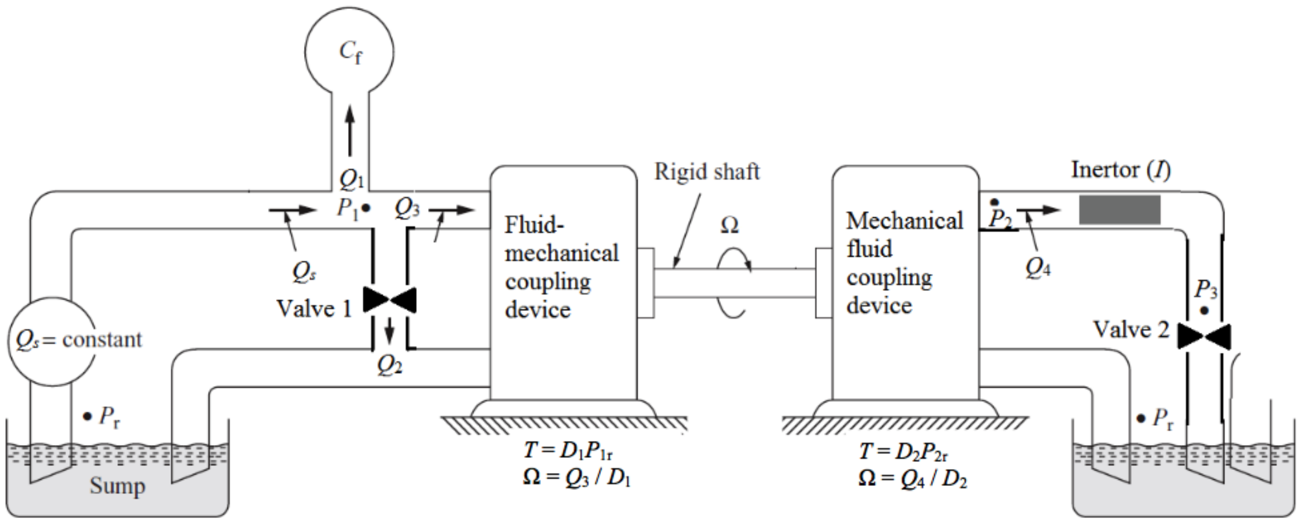


Figure 3: A simplified turbocharger

The exhaust gas flow Q_s which is connected to a capacitor (with capacitance C_f) is used to rotate the turbocharger compressor through a series of coupling devices where the first coupling device has the relation of

$$\begin{aligned} T &= D_1 P_{1r} \\ \Omega &= Q_3/D_1 \end{aligned} \tag{67}$$

and the second coupling device has the relation of

$$\begin{aligned} T &= D_2 P_{2r} \\ \Omega &= Q_4/D_2 \end{aligned} \tag{68}$$

In these equations D_1 and D_2 are coupling constants. For a simplification, let the valve 1 satisfy

$$Q_2 = k_1 P_{1r}, \tag{69}$$

where k_1 is a constant. The turbocharger compressor is connected to a load and an inertor, where, for simplicity, the valve 2 satisfies

$$P_{3r} = k_2 Q_4, \tag{70}$$

where k_2 is a constant, and the inertor has inertance of I.

(a) (1 point) Which are A-Type and T-Type elements and their related variables?

Solution: The elements are the fluid capacitor and the inertor. The related variables are the pressure P_{1r} and the flow Q_4 respectively.

(b) (1 point) Write down the constitutive equations of each of the energy storing elements of the simplified turbocharger.

Solution: The constitutive equation of the fluid capacitor is $C\frac{dP_{1r}}{dt} = Q_1$ while the constitutive equation of the inertor is $I\frac{dQ_4}{dt} = P_{32}$.

(c) (4 points) Assume that the input u is given by the flow source Q_s and the measured output y is the flow Q_4, write down the state space equation describing the turbocharger.

Solution: Since there are two energy storing elements corresponding to the capacitor pressure P_{1r} and the fluid flow into the into the inertor Q_4, as indicated above we have:

- For the capacitor:

$$
\begin{aligned}
C_f \frac{d}{dt} P_{1r}(t) &= Q_1(t) = Q_s(t) - Q_2(t) - Q_3(t) \\
&= Q_s(t) - k_1 P_{1r}(t) - D_1 \Omega(t) \\
&= Q_s(t) - k_1 P_{1r}(t) - {}^{D_1}\!/_{D_2} Q_4(t)
\end{aligned} \tag{71}
$$

  and therefore

$$
\frac{d}{dt} P_{1r}(t) = \frac{1}{C_f} \left(Q_s(t) - k_1 P_{1r}(t) - {}^{D_1}\!/_{D_2} Q_4(t) \right) \tag{72}
$$

- For the inertor we have

$$
\begin{aligned}
I \frac{d}{dt} Q_4(t) &= P_{23}(t) = P_{2r}(t) - P_{3r}(t) \\
&= {}^{T(t)}\!/_{D_2} - k_2 Q_4(t) \\
&= {}^{D_1}\!/_{D_2} P_{1r}(t) - k_2 Q_4(t)
\end{aligned} \tag{73}
$$

  and therefore

$$
\frac{d}{dt} Q_4(t) = \frac{1}{I} \left({}^{D_1}\!/_{D_2} P_{1r}(t) - k_2 Q_4(t) \right). \tag{74}
$$

From these equations and the fact that $u = Q_s$, $y = Q_4$ we can write the state equation

$$
\begin{aligned}
\begin{bmatrix} \dot{P}_{1r} \\ \dot{Q}_4 \end{bmatrix} &= \begin{bmatrix} -{}^{k_1}\!/_{C_f} & -{}^{D_1}\!/_{C_f D_2} \\ {}^{D_1}\!/_{I D_2} & -{}^{k_2}\!/_{I} \end{bmatrix} \begin{bmatrix} P_{1r} \\ Q_4 \end{bmatrix} + \begin{bmatrix} {}^{1}\!/_{C_f} \\ 0 \end{bmatrix} u \\
y &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} P_{1r} \\ Q_4 \end{bmatrix}
\end{aligned} \tag{75}
$$

(d) (4 points) Does the state-space equation that you have found have the same input-output behavior (or transfer function) of the following state-space equation? Motivate your answer.

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}0 & 1\\ -\dfrac{D_1^2}{IC_fD_2}-\dfrac{k_1k_2}{IC_f} & -\dfrac{k_1}{C_f}-\dfrac{k_2}{I}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u\\ y &= \begin{bmatrix}\dfrac{D_1}{IC_fD_2} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\end{aligned} \tag{76}$$

Solution: The corresponding transfer function is given by

$$\begin{aligned}G(s)=\frac{Y(s)}{U(s)}=C(sI-A)^{-1}B &= \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}s+{}^{k_1}/_{C_f} & {}^{D_1}/_{C_fD_2}\\ -{}^{D_1}/_{ID_2} & s+{}^{k_2}/_{I}\end{bmatrix}^{-1}\begin{bmatrix}{}^{1}/_{C_f}\\ 0\end{bmatrix}\\ &= \frac{{}^{D_1}/_{IC_fD_2}}{s^2+\left({}^{k_1}/_{C_f}+{}^{k_2}/_{I}\right)s+{}^{k_1k_2}/_{IC_f}+{}^{D_1^2}/_{IC_fD_2}}.\end{aligned} \tag{77}$$

Note that the state-space equation given in the question corresponds to the canonical controllable form of the transfer function. Therefore BOTH state-space representations have the same input-output behavior.